INVERSE TRIGONOMETRIC FUNCTIONS

A function $f : A \to B$ is invertible if and only if it is a bijective. If $y = f(x) \Leftrightarrow x = f^{-1}(y)$, is known as x is equal to *f* inverse y.

There are six inverse functions viz. $\sin^{-1} x, \cos^{-1} x, \tan^{-1} x, \operatorname{cosec}^{-1} x, \sec^{-1} x$ *and* $\cot^{-1} x$.

Principal value of an Inverse function

The ranges of the inverse trigonometric functions are called principal values of the inverse function. The principal value of $\sin^{-1} x$, $\tan^{-1} x$ and $\operatorname{cosec}^{-1} x$ are the angles that lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ and the principal value of $\cos^{-1} x$, $\sec^{-1} x$ and $\cot^{-1} x$ are the angles that lie between 0 and π.

Domain and Range of Inverse T-functions

Function	Domain	Range
$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$(-\infty, \infty)$ *or* R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\operatorname{cosec}^{-1} x$	$x \le -1$ *or* $x \ge 1$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$\sec^{-1} x$	$x \le -1$ *or* $x \ge 1$	$[[0, \pi]] - \left\{\frac{\pi}{2}\right\}$
$\cot^{-1} x$	$(-\infty, \infty)$ *or* R	$(0, \pi)$

Properties of Inverse T-functions

If $\theta = \sin^{-1} x \Rightarrow \sin\theta = x$, where $-1 \le x \le 1$ *and* $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

For example, if $\theta = \sin^{-1} 1 \Rightarrow \sin\theta = 1 \Rightarrow \sin\theta = \sin\frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{2}$

Property 1:

1. $\sin^{-1}(\sin x) = \sin(\sin^{-1} x) = x$
2. $\cos^{-1}(\cos x) = \cos(\cos^{-1} x) = x$
3. $\tan^{-1}(\tan x) = \tan(\tan^{-1} x) = x$
4. $\text{cosec}^{-1}(\text{cosec}\, x) = \text{cosec}(\text{cosec}^{-1} x) = x$
5. $\sec^{-1}(\sec x) = \sec(\sec^{-1} x) = x$
6. $\cot^{-1}(\cot x) = \cot(\cot^{-1} x) = x$

Property 2:

1. $\sin^{-1} x = \text{cosec}^{-1}\left(\frac{1}{x}\right), |x| \le 1$
2. $\cos^{-1} x = \sec^{-1}\left(\frac{1}{x}\right), |x| \le 1$
3. $\tan^{-1} x = \cot^{-1}\left(\frac{1}{x}\right), x \in R$
4. $\text{cosec}^{-1} x = \sin^{-1}\left(\frac{1}{x}\right), |x| \ge 1$
5. $\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right), |x| \ge 1$
6. $\cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right), x > 0$
7. $\cot^{-1} x = \begin{cases} \pi + \tan^{-1}\left(\frac{1}{x}\right), & x < 0 \\ \tan^{-1}\left(\frac{1}{x}\right), & x > 0 \end{cases}$

Property 3:

1. $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, |x| \le 1 \text{ or } x \in [-1, 1]$
2. $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, x \in R$
3. $\text{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}, |x| \ge 1 \text{ or } x \in R - (-1, 1)$

Property 4:

1. $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$, *if* $-1 \le x, y \le 1$ *and* $x^2 + y^2 \le 1$

2. $\sin^{-1} x - \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$, *if* $-1 \le x, y \le 1$ *and* $x^2 + y^2 \le 1$

3. $\cos^{-1} x + \cos^{-1} y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$, *if* $-1 \le x, y \le 1$ *and* $x + y \ge 0$

4. $\cos^{-1} x - \cos^{-1} y = \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)$, *if* $-1 \le x, y \le 1$ *and* $y - x \ge 0$

5. $$\tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1}\left(\frac{x+y}{1-xy}\right) & \text{if } x>0, y>0 \text{ and } xy<1 \\ \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right) & \text{if } x>0, y>0 \text{ and } xy>1 \\ \tan^{-1}\left(\frac{x+y}{1-xy}\right) - \pi & \text{if } x<0, y<0 \text{ and } xy>1 \end{cases}$$

6. $$\tan^{-1} x - \tan^{-1} y = \begin{cases} \tan^{-1}\left(\frac{x-y}{1+xy}\right) & \text{if } x>0, y<0 \text{ and } xy>-1 \\ \pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right) & \text{if } x>0, y<0 \text{ and } xy<-1 \\ \tan^{-1}\left(\frac{x-y}{1+xy}\right) - \pi & \text{if } x<0, y>0 \text{ and } xy<-1 \end{cases}$$

Property 5: (Conversion formulae)

1. $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1} x, \; x \in [-1, 1]$

2. $\cos^{-}\left(\frac{1-x^2}{1+x^2}\right) = 2\tan^{-1} x, \; x \in [0, \infty)$

3. $\tan^{-1}\left(\frac{2x}{1-x^2}\right) = 2\tan^{-1} x, \; x \in (-1, 1)$

4. $3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$ *if* $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

5. $3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$ *if* $x \in \left[\frac{1}{2}, 1\right]$

6. $3\tan^{-1} x = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$ *if* $x \in \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

7. $\cos^{-1} x = \sin^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

8. $\sin^{-1} x = \cos^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

9. $\tan^{-1} x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \sec^{-1}\left(\sqrt{1+x^2}\right) = \operatorname{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$

10. $2\sin^{-1} x = \sin^{-1}\left(2x\sqrt{1+x^2}\right)$

11. $2\cos^{-1} x = \cos^{-1}\left(2x^2 - 1\right)$

Note:

1. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$, *then* $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$
2. If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, *then* $x^2 + y^2 + z^2 + 2xyz = 1$
3. If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, *then* $x + y + z = xyz$
4. If $\cot^{-1}x + \cot^{-1}y + \cot^{-1}z = \frac{\pi}{2}$, *then* $x + y + z = xyz$

Tips ☞

1. $(\sin x)^{-1} \neq \sin^{-1} x$
2. Range is same as principal value range of the inverse circular function.
3. To learn this chapter well, you should thorough in Trigonometric functions especially, compound angles, reduction formulae, multiple and sub multiple angles (Portions available in **Formula Master Part I** of the author)