9. DIFFERENTIAL EQUATION

An equation involving differential coefficient is known as differential equation. In other words, an equation involving derivative or derivatives of dependent variable(s) with respect to independent variable is known as differential equation.

Degree

Degree of a differential equation is the degree of the highest derivative occurs in it, when differential coefficients are made free from radicals and fractions.

Order

Order of a differential equation is the order of the highest derivative occurs in it.

E.g.:

Differential equation	Degree	Order
$\frac{d^2y}{dx^2} = 4x + 1$	1	2
$xy = x\frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$	2	1
$\frac{d^3y}{dx^3} - 4x\left(\frac{d^2y}{dx^2}\right)^2 + 2y\frac{dy}{dx} = 0$	1	3
$\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2} = 5\frac{d^2y}{dx^2}$	2	2
$\left(\frac{d^2y}{dx^2}\right)^3 + 3\sin\left(\frac{dy}{dx}\right) = 0$	2	Not defined.

Formation of a differential equation

To form a differential equation from a given equation with n constants, differentiate n times and eliminate the constants using the n equations (equations formed in each step of differentiation) with the given equation.

Working rule:

i. Write the given equation
ii. Count the number of arbitrary constant(s)
iii. Differentiate the given equation successively as many times as the number of arbitrary constant(s)
iv. Eliminate the arbitrary constant(s) by using the given equation obtained in (iii) or the equations in (i) and (ii) and so is the required differential equation.

1. Find the differential equation of all straight lines in the XY plane.

Let the equation is $y = mx + c$

Differentiating y w.r.t. x

$$\frac{dy}{dx} = m \times 1 + 0$$

$$\frac{dy}{dx} = m$$

Differentiating again w.r.t. x

$\frac{d^2y}{dx^2} = 0$ is the required differential equation.

2. Find the differential equation of all straight lines passing through the origin.

Let the equation is $y = mx$ .... (1)

Differentiating y w.r.t. x

$$\frac{dy}{dx} = m \times 1 = m$$

in (1) we have, $y = \frac{dy}{dx} x \Rightarrow x\frac{dy}{dx} - y = 0$ is the required differential equation.

3. Find the differential equation of all circles passing through the origin and centre lies on the x axis.

Equation is $(x-a)^2 + y^2 = a^2$

Differentiating w.r.t. x

$2\,(x-a) + 2y\frac{dy}{dx} = 0 \Rightarrow x - a + y\frac{dy}{dx} = 0 \Rightarrow y\frac{dy}{dx} - x + a = 0$ is the differential equation.

Solution or primitive of a differential equation

Solution is a functional relation between the variable(s) involved and it satisfies the given differential equation. A solution of a differential equation is called general solution or complete solution.

A general solution may contain one or more arbitrary constant(s) and it depends upon the order of the differential equation.

E.g.: $\frac{dy}{dx}+e^{x-y}=0$

$\frac{dy}{dx}=-e^{x-y}=-\frac{e^x}{e^y}$

$e^y dy=-e^x dx$

integrating with respect to x we have,

$\int e^y dy=-\int e^x dx$

$e^y=-e^x+C \Rightarrow e^x+e^y=C$ is known as general solution of the differential equation.

Particular Solution.

The solution obtained by particular value(s) of the arbitrary constant(s) in the general solution is called particular solution. In a particular solution, there is no arbitrary constant(s).

E.g.: $\frac{dy}{dx}+e^{x-y}=0,\ y(0)=0$

$\frac{dy}{dx}=-e^{x-y}=-\frac{e^x}{e^y}$

$e^y dy=-e^x dx$

integrating with respect to x we have,

$\int e^y dy=-\int e^x dx$

$e^y=-e^x+C \Rightarrow e^x+e^y=C$

$when\ x=0,\ y=0,\ we\ have\ e^0+e^0=C \Rightarrow C=1+1=2$

$\therefore e^x+e^y=2$ is the particular solution.

Methods of finding the solution

1. Variable Separable: A differential equation of the form $\boldsymbol{Mdx+Ndy=0}$, where M is a function of x and N is a function of y. Then solution of such differential equation is obtained by integrating on both sides w.r.t. the specified variable.

   E.g.: Solve $\sec^2 x\tan y dx+\sec^2 y\tan x dy=0$

$$\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$$

$$\textit{dividing by } \tan x \tan y$$

$$\frac{\sec^2 x}{\tan x} dx + \frac{\sec^2 y}{\tan y} dy = 0 \textit{ is in VS}$$

integrating we have,

$$\int \frac{\sec^2 x}{\tan x} dx + \int \frac{\sec^2 y}{\tan y} dy = C$$

$$\log|\tan x| + \log|\tan y| = \log C$$

$$\log|\tan x \tan y| = \log C$$

$\Rightarrow \tan x \tan y = C$ is the solution.

2. Solve $\dfrac{dy}{dx} + \sqrt{\dfrac{1-x^2}{1-y^2}} = 0$

$$\frac{dy}{dx} = \sqrt{\frac{1-x^2}{1-y^2}} = \frac{\sqrt{1-x^2}}{\sqrt{1-y^2}}$$

$\sqrt{1-y^2}\,dy = \sqrt{1-x^2}\,dx$ is in variable separable.

Integrating we have, $\int \sqrt{1-y^2}\,dy = \int \sqrt{1-x^2}\,dx$

$$\frac{2}{3}\left(1-y^2\right)^{\frac{3}{2}} \times \frac{1}{-2y} = \frac{2}{3}\left(1-x^2\right)^{\frac{3}{2}} \times \frac{1}{-2x} + C$$

$$\left(1-y^2\right)^{\frac{3}{2}} \times \frac{1}{-y} = \left(1-x^2\right)^{\frac{3}{2}} \times \frac{1}{-x} + C$$

$$\frac{\left(1-x^2\right)^{\frac{3}{2}}}{x} - \frac{\left(1-y^2\right)^{\frac{3}{2}}}{y} = C$$

3. Solve $\dfrac{dy}{dx} + \sqrt{\dfrac{1-y^2}{1-x^2}} = 0$

$$\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$\dfrac{dy}{\sqrt{1-y^2}} = -\dfrac{dx}{\sqrt{1-x^2}}$ is in variable separable.

Integrating we have,

$$\int\frac{dy}{\sqrt{1-y^2}}+\int\frac{dx}{\sqrt{1-x^2}}=C$$

$$\sin^{-1}y+\sin^{-1}x=C$$

Homogeneous differential equation

A differential equation is of the form $\frac{dy}{dx}=\frac{f(x,y)}{\phi(x,y)}$, where $f(x,y)$ and $\phi(x,y)$ are functions of the same degree is known as homogeneous differential equation.

Working rule:

Put $y=vx$

Differentiating w.r.t. x

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

Then,

$$v+x\frac{dv}{dx}=\frac{f(x,vx)}{\phi(x,vx)}=x\frac{dv}{dx}=\frac{f(1,v)}{\phi(1,v)}-v\Rightarrow x\frac{dv}{dx}=\frac{f(1,v)-v\phi(1,v)}{\phi(1,v)}$$

$\frac{\phi(1,v)}{f(1,v)-v\phi(1,v)}dv=\frac{dx}{x}$ is in variable separable and hence can be evaluated.

E.g.: Check whether the following differential equation is homogeneous, if so, solve it:

1. $y'=\frac{x+y}{x}$

$$\Rightarrow\frac{dy}{dx}=\frac{x+y}{x}=\frac{x\left(1+\frac{y}{x}\right)}{x}=1+\frac{y}{x}$$

Since the R.H.S is of the form $g\left(\frac{y}{x}\right)$, and so it is a homogeneous of degree zero.

$$put\ y=vx\Rightarrow v=\frac{y}{x}$$

$$and\ \frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$v+x\frac{dv}{dx}=1+v$$

$$x\frac{dv}{dx}=1$$

$dv = \frac{dx}{x}$, is in variable separable.

$$\therefore \int dv = \int \frac{dx}{x}$$

$$v = \log|x| + C$$

$$\frac{y}{x} = \log|x| + C \Rightarrow y = x\log|x| + Cx$$

$$Hence, y = x\log|x| + C(say)$$

2. $\left(1+e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0$

$$e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = -\left(1+e^{\frac{x}{y}}\right)dx$$

$$\frac{dx}{dy} = \frac{e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{-\left(1+e^{\frac{x}{y}}\right)} = \frac{e^{\frac{x}{y}}\left(\frac{x}{y}-1\right)}{\left(1+e^{\frac{x}{y}}\right)}$$

The RHS is a function of the form $g\left(\frac{x}{y}\right)$, and is a homogeneous function of degree zero.

$$\frac{dx}{dy} = \frac{e^{\frac{x}{y}}\left(\frac{x}{y}-1\right)}{\left(1+e^{\frac{x}{y}}\right)} \quad \text{.............(1)}$$

$put\ v = \frac{x}{y}$

$$\Rightarrow x = vy\ and\ \frac{dx}{dy} = v + y\frac{dv}{dy}$$

(1) becomes

$$v+y\frac{dv}{dy}=\frac{e^v(v-1)}{(1+e^v)}$$

$$y\frac{dv}{dy}=\frac{e^v(v-1)}{(1+e^v)}-v=\frac{e^v(v-1)-v(1+e^v)}{(1+e^v)}$$

$$y\frac{dv}{dy}=\frac{ve^v-e^v-v-ve^v}{(1+e^v)}=\frac{-(e^v+v)}{(1+e^v)}$$

$$\frac{(e^v+1)}{(e^v+v)}dv=-\frac{dy}{y}\quad is\ in\ VS$$

$$\int\frac{(e^v+1)}{(e^v+v)}dv=-\int\frac{dy}{y}$$

$$\log\left|e^v+v\right|=-\log|y|+\log C$$

$$\log\left|e^{\frac{x}{y}}+\frac{x}{y}\right|+\log|y|=\log C$$

$$\log\left|\frac{ye^{\frac{x}{y}}+x}{y}\right|y=\log C\Rightarrow\left|\frac{ye^{\frac{x}{y}}+x}{y}\right|y=C$$

$$\left(\frac{ye^{\frac{x}{y}}+x}{y}\right)y=\pm C\Rightarrow ye^{\frac{x}{y}}+x=C_1$$

$ye^{\frac{x}{y}}+x=C$ is the required solution.

Linear differential equation

A differential equation is of the form $\frac{dy}{dx}+Py=Q$, where P and Q are constants or functions of x, is known as linear differential equation.

Note: In a linear differential equation, the coefficient of $\frac{dy}{dx}$ must be 1. Otherwise divide the equation by the coefficient of $\frac{dy}{dx}$ to make the coefficient 1. Then,

1. Let P = a constant or a function of x and Q = a constant or a function of x.
2. Find $\int P dx$
3. Find $e^{\int P dx}$
4. Find $Qe^{\int p dx}$
5. Find $\int Qe^{\int p dx} dx$
6. Solution is $ye^{\int P dx} = \int Qe^{\int p dx} dx + C$

Note1: Here the quantity $e^{\int P dx}$ is known as integrating factor.

Note2: $e^{\log[f(x)]} = f(x)$ *and* $\log e^{f(x)} = f(x)$

E.g.:

1) Solve: $\frac{dy}{dx} + y\cot x = \sin x$

Let $P = \cot x$ and $Q = \sin x$

$$\int P dx = \int \cot x dx = \log\sin x$$

$$e^{\int P dx} = e^{\log\sin x} = \sin x$$

$$Qe^{\int p dx} = \sin x \sin x = \sin^2 x$$

$$\int Qe^{\int p dx} dx = \int \sin^2 x dx = \int \frac{1+\cos x}{2} dx = \frac{1}{2}(x + \sin x)$$

Solution is $ye^{\int p dx} = \int Qe^{\int p dx} dx + C$

$$y\sin x = \frac{1}{2}(x + \sin x) + C$$

2) Solve the differential equation: $\cos x \frac{dy}{dx} + y\sin x = \tan^2 x$

To make the coefficient of $\frac{dy}{dx}$, one, dividing by $\cos x$

$$\frac{dy}{dx} + y\frac{\sin x}{\cos x} = \frac{\tan^2 x}{\cos x}$$

$$\frac{dy}{dx} + y\tan x = \tan^2 x\sec x$$

$$p = \tan x;\ Q = \tan^2 x\sec x$$

$$\int p\,dx = \int \tan x\,dx = \log|\sec x|$$

$$e^{\int p\,dx} = e^{\log|\sec x|} = \sec x$$

$$Qe^{\int p\,dx} = \tan^2 x\sec x\sec x$$

$$\int Qe^{\int p\,dx}dx = \int \tan^2 x\sec^2 x\,dx$$

Put $\tan x = u \Rightarrow \sec^2 x\,dx = du$

$$\int Qe^{\int p\,dx}dx = \int u^2\,du = \frac{u^3}{3} + C = \frac{\tan^3 x}{3} + C$$

Solution is $ye^{\int p\,dx} = \int Qe^{\int p\,dx}dx + C$

$$y\sec x = \frac{\tan^3 x}{3} + C$$

3) Find the solution of the differential equation: $\left(x+3y^2\right)\frac{dy}{dx} = y\ \left(y>0\right)$

$$\left(x+3y^2\right)\frac{dy}{dx} = y\ \left(y>0\right)$$

$$\frac{dy}{dx} = \frac{y}{\left(x+3y^2\right)} \Rightarrow \frac{dx}{dy} = \frac{x+3y^2}{y}$$

$\frac{dx}{dy} - \frac{1}{y}x = 3y$ is a linear differential equation in y.

$$p = -\frac{1}{y};\ Q = 3y$$

$$\frac{dx}{dy} - \frac{1}{y}x = 3y$$

$$\int p\,dy = -\int \frac{1}{y}\,dy = -\log|y| = \log\left|\frac{1}{y}\right|$$

$$e^{\int p\,dy} = e^{\log\left|\frac{1}{y}\right|} = \frac{1}{y}$$

$$\int Qe^{\int p\,dy} = \int 3y \times \frac{1}{y}\,dy = 3\int dy = 3y$$

Solution is $xe^{\int p\,dy} = \int Qe^{\int p\,dy}dy + C$ [note this]

$$x.\frac{1}{y} = 3y + C \Rightarrow x = 3y^2 + Cy$$

$x = 3y^2 + C(say)$ is the required solution.

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